INTERNATIONAL MATHEMATICS
TOURNAMENT OF TOWNS

SENIOR PAPER: YEARS 11,12

Tournament 42, Northern Spring 2021 (O Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. (a) A convex pentagon is partitioned into three triangles by non-intersecting diagonals. Is it possible for centroids of these triangles to lie on a common straight line? (2 points)

   (b) The same question for a non-convex pentagon. (2 points)

2. (a) Maria has a balance scale that can indicate which of its pans is heavier or whether they have equal weight. She also has 4 weights that look the same but have masses of 1000, 1002, 1004 and 1005 grams. Can Maria determine the mass of each weight in 4 weighings? The weights for a new weighing may be picked when the results of the previous ones are known. (2 points)

   (b) The same question when the left pan of the scale is lighter by 1 gram than the right one, so the scale indicates equality when the mass on the left pan is heavier by 1 gram than the mass on the right pan. (2 points)

3. For which n is it possible that a product of n consecutive positive integers is equal to a sum of n consecutive (not necessarily the same) positive integers? (5 points)

4. It is well-known that a quadratic equation has no more than 2 roots. Is it possible for the equation $\lfloor x^2 \rfloor + px + q = 0$ with $p \neq 0$ to have more than 100 roots? (By $\lfloor x^2 \rfloor$ we denote the largest integer not greater than x^2.) (5 points)

5. Let O be the circumcentre of an acute triangle ABC. Let M be the midpoint of AC. The straight line BO intersects the altitudes AA_1 and CC_1 of the triangle ABC at the points H_a and H_c respectively. The circumcircles of the triangles BH_aA and BH_cC have a second point of intersection K. Prove that K lies on the straight line BM. (6 points)